

# THE GROUP OF AUTOMORPHISMS OF THE PENTABLOCK

ŁUKASZ KOSIŃSKI

**ABSTRACT.** Answering the question posed in [Ag-Ly-Yo] we determine the group of holomorphic automorphisms of the pentablock. The geometry of the pentablock (linear convexity, being an analytic retract of special convex domains) is also studied.

The pentablock is a domain appearing naturally in  $\mu$ -synthesis problems. It is a subdomain of  $\mathbb{C}^3$  denoted by  $\mathcal{P}$  and defined as an image of the classical  $2 \times 2$  Cartan domain of the first type  $\mathcal{R}_I = \{z \in \mathbb{C}^{2 \times 2} : \|z\| < 1\}$  under the mapping

$$\pi : z = (z_{ij}) \mapsto (z_{12}, \operatorname{tr} z, \det z).$$

Some computations in [Ag-Ly-Yo] allow us to observe that the pentablock is a Hartogs domain in  $\mathbb{C}^3$  over a  $\mathbb{C}$ -convex domain  $\mathbb{G}_2 = \{(s, p) \in \mathbb{C}^2 : |s - \bar{s}p| + |p|^2 < 1\}$  (for the  $\mathbb{C}$ -convexity of the symmetrized bidisc see [Ni-Pf-Zw]). More precisely,

$$(1) \quad \mathcal{P} = \{(a, s, p) \in \mathbb{D} \times \mathbb{G}_2 : |a|^2 < e^{-u(s, p)}\},$$

where

$$u(s, p) = -2 \log \left| 1 - \frac{\frac{1}{2}s\bar{\beta}}{1 + \sqrt{1 - |\beta|^2}} \right|$$

and  $\beta = \frac{s - \bar{s}p}{1 - |p|^2}$ .

An immediate consequence of descriptions mentioned above is the fact that  $\mathcal{P}$  is an  $(1, 0, 0)$ - and  $(0, 1, 2)$ -balanced domain. This means that  $(\lambda a, s, p) \in \mathcal{P}$  and  $(a, \lambda s, \lambda^2 p) \in \mathcal{P}$  for any  $(a, s, p) \in \mathcal{P}$  and  $\lambda \in \mathbb{D}$ .

The main aim of this note is to determine the group of holomorphic automorphisms of  $\mathcal{P}$ . We shall also investigate geometric properties of the pentablock. In particular, we shall prove that the pentablock is not an analytic retract of the open unit ball of any  $\mathcal{J}^*$  algebra of finite rank. These results answer questions posed in [Ag-Ly-Yo].

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Probably, some calculations may show that the function  $u$  appearing in (1) is  $\mathbb{C}$ -convex. Since these calculations seem to be tedious and difficult we shall show this fact avoiding them. Of course, using  $\mathbb{C}$ -convexity of the function  $u$  number of geometric properties of the pentablock may be derived immediately.

Here and throughout the paper  $\mathbb{D}$  denotes the unit disc in the complex plane, additionally by  $\mathbb{T}$  we shall denote the unit circle.

Recall that the special subgroup of the group of automorphisms of the pentablock was constructed in [Ag-Ly-Yo]. More precisely, it was shown that any mapping of the form

$$(2) \quad f_{\omega, \nu}(a, \lambda_1 + \lambda_2, \lambda_1 \lambda_2) = \left( \frac{\omega \eta (1 - |\alpha|^2) a}{1 - \bar{\alpha}(\lambda_1 + \lambda_2) + \bar{\alpha}^2 \lambda_1 \lambda_2}, \nu(\lambda_1) + \nu(\lambda_2), \nu(\lambda_1) \nu(\lambda_2) \right),$$

where  $\omega \in \mathbb{T}$  and  $\nu$  is a Möbius function of the form  $\nu(\lambda) = \eta \frac{\lambda - \alpha}{1 - \bar{\alpha} \lambda}$ , is an automorphism of the pentablock.

In what follows we shall show that above mappings form the whole group of automorphisms of the pentablock:

**Theorem 1.**

$$\text{Aut}(\mathcal{P}) = \{f_{\omega, \nu} : \omega \in \mathbb{T}, \nu \text{ is a Möbius function}\}.$$

*In particular, the pentablock is inhomogeneous.*

The crucial tools used in deducing the result presented above involve elementary properties of biholomorphic mappings between quasi-balanced domains, the description of proper holomorphic selfmappings of the symmetrized bidisc and the classical Cartan theorem.

First of all recall that a simple modification of a Bell's argument (see [Kos]) allows us to observe that any proper holomorphic mapping between quasi-balanced domains whose Minkowski functionals are continuous (this assumption is clearly satisfied by the pentablock) extends holomorphically past neighborhoods of closure of domains. Applying this fact to an automorphism  $\varphi$  of the pentablock and to its inverse  $\varphi^{-1}$  we find that:

**Lemma 2.** *Any automorphism of the pentablock extends to a biholomorphic mapping between some neighborhoods of  $\overline{\mathcal{P}}$ .*

In the sequel we shall make use of the geometry of the symmetrized bidisc. Recall that it may be also described as  $\mathbb{G}_2 = \{(\lambda_1 + \lambda_2, \lambda_1 \lambda_2) : \lambda_1, \lambda_2 \in \mathbb{D}\}$ . By  $\Sigma$  we shall denote its royal variety, i.e.  $\Sigma = \{(2\lambda, \lambda^2) :$

$\lambda \in \bar{\mathbb{D}}\}$  and by  $\partial_s \mathbb{G}_2$  its Shilov boundary. Recall that  $\partial_s \mathbb{G}_2 = \{(\lambda_1 + \lambda_2, \lambda_1 \lambda_2) : \lambda_1, \lambda_2 \in \mathbb{T}\}$ .

Let us list two elementary properties of the topological boundary of the pentablock:

1) any point  $x$  of  $\partial_1 \mathcal{P} := \{(a, s, p) \in \bar{\mathbb{D}} \times \mathbb{G}_2 : |a|^2 = e^{-u(s,p)}\}$  is a smooth point of  $\partial \mathcal{P}$ . Moreover, the rank of the Levi form of a defining function of  $\partial \mathcal{P}$  at the point  $x$  restricted to the complex tangent space is equal to 1. Of course this fact may be checked directly. But to avoid tedious computations it suffices to observe that  $\partial_1 \mathcal{P}$  is not Levi flat, as  $u$  is not a pluriharmonic function. Moreover,  $x$  is not a point of strict pseudoconvexity. To show it, take  $z \in \partial \mathcal{R}_I$  such that  $\pi(z) = x$ . By the singular value decomposition theorem, there are unitary matrices  $U$  and  $V$  and  $\zeta_0 \in \bar{\mathbb{D}}$  such that  $z = U \begin{pmatrix} 1 & 0 \\ 0 & \zeta_0 \end{pmatrix} V$ . Since  $x \in \bar{\mathbb{D}} \times \mathbb{G}_2$ ,

$\zeta_0 \in \mathbb{D}$ . Then  $t \mapsto \pi(U \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} V)$  is a non-trivial analytic disc in  $\bar{\mathcal{P}}$  passing through  $x$ . Since  $u$  is subharmonic (see e.g. Proposition 3), we find that this analytic disc lies in  $\partial \mathcal{P}$ .

2)  $\partial_2 \mathcal{P} := \{(a, s, p) \in \mathbb{D} \times \bar{\mathbb{G}}_2 : (s, p) \in \partial \mathbb{G}_2 \setminus \partial_s \mathbb{G}_2, |a|^2 < e^{-u(s,p)}\}$  is a Levi flat part of  $\partial \mathcal{P}$ . To observe it suffices to note that  $e^{-u(s,p)} > 0$  if  $(s, p) \in \partial \mathbb{G}_2 \setminus \Sigma$ . This may be derived immediately from another description of the pentablock obtained in [Ag-Ly-Yo]:  $\mathcal{P} = \{(a, \lambda_1 + \lambda_2, \lambda_1 \lambda_2) : |a| < \frac{1}{2}|1 - \lambda_1 \bar{\lambda}_2| + \frac{1}{2}(1 - |\lambda_1|^2)^{1/2}(1 - |\lambda_2|^2)^{1/2}, \lambda_1, \lambda_2 \in \mathbb{D}\}$ .

Now we are ready to find a description of the group of automorphism of the pentablock.

*Proof of Theorem 1.* Let us take  $\varphi \in \text{Aut}(\mathcal{P})$ .

Step 1. Consider the mapping

$$\Psi : \mathbb{G}_2 \ni (s, p) \mapsto (\varphi_2(0, s, p), \varphi_3(0, s, p)) \in \mathbb{G}_2.$$

As mentioned above,  $\Psi$  extends holomorphically past  $\bar{\mathbb{G}}_2$ . Moreover, any point  $(0, s, p)$ , where  $(s, p) \in \partial \mathbb{G}_2 \setminus \partial_s \mathbb{G}_2$ , lies in a Levi flat part of the boundary of the pentablock. Thus,  $\varphi(0, s, p)$  lies in a Levi flat part as well. Therefore, a simply continuity argument shows that  $\Psi$  maps  $\partial \mathbb{G}_2$  into  $\partial \mathbb{G}_2$ . In particular,  $\Psi$  is a proper holomorphic self-mapping of  $\mathbb{G}_2$ .

Step 2. It follows from [Edi-Zwo] that there is a Blaschke product  $b$  such that

$$\Psi(\lambda_1 + \lambda_2, \lambda_1 \lambda_2) = (b(\lambda_1) + b(\lambda_2), b(\lambda_1)b(\lambda_2)) \quad \lambda_1, \lambda_2 \in \mathbb{D}.$$

A direct consequence of this fact is that  $\Psi$  preserves the royal variety of the symmetrized bidisc, i.e.  $\Psi(\Sigma) = \Sigma$ .

Note that a point  $(a, s, p) \in \bar{\mathbb{D}} \times (\Sigma \cap \partial\mathbb{G}_2)$  lies in  $\bar{\mathcal{P}}$  if and only if  $a = 0$ . This, in particular, means that  $\lambda \mapsto \varphi_1(0, 2\lambda, \lambda^2)$  vanishes on  $\mathbb{T}$ , whence  $(s, p) \mapsto \varphi_1(0, s, p)$  vanishes on  $\Sigma$ . Therefore,

$$(3) \quad \varphi(0, 2\lambda, \lambda^2) = (0, 2b(\lambda), b(\lambda)^2), \quad \lambda \in \mathbb{D}.$$

Moreover, there is a holomorphic mapping  $\alpha_1$  on  $\mathbb{G}_2$  such that  $\varphi_1(0, s, p) = (s^2 - 4p)\alpha_1(s, p)$  for  $(s, p) \in \mathbb{G}_2$ .

Step 3. Applying (3) to  $\varphi^{-1}$  we get immediately that  $b$  is a Möbius function. Composing  $\varphi$  with an automorphism of the form (3) we may assume that  $b(\lambda) = \lambda$ ,  $\lambda \in \mathbb{D}$ .

For now we have shown that, up to a composition with an automorphism of the form (3),

$$\varphi(0, s, p) = ((s^2 - 4p)\alpha_1(s, p), s, p), \quad (s, p) \in \mathbb{G}_2.$$

In particular,

$$\begin{aligned} \varphi(a, s, p) = & ((s^2 - 4p)\alpha_1(s, p) + \alpha_2 a + a\alpha_3(a, s, p), \\ & \beta_1 a + s + a\beta_2(a, s, p), \gamma_1 a + p + a\gamma_2(a, s, p)) \end{aligned}$$

for  $(a, s, p) \in \mathcal{P}$ , where  $\alpha_2, \beta_1, \gamma_1 \in \mathbb{C}$  and  $\alpha_3, \beta_2, \gamma_2 \in \mathcal{O}(\mathcal{P})$  vanish at the origin.

Step 4. Now we shall make use of the fact that the pentablock is  $(2, 1, 2)$ - and  $(1, 1, 2)$ -balanced. For  $m = (m_1, m_2, m_3) \in \mathbb{N}^3$  and  $\lambda \in \mathbb{D}$  denote the action on  $\mathbb{C}^3$

$$m_\lambda \cdot x = (\lambda^{m_1} x_1, \lambda^{m_2} x_2, \lambda^{m_3} x_3).$$

Let  $m = (2, 1, 2)$ . Note that for any unimodular  $\lambda$  the mapping

$$\varphi_\lambda : (a, s, p) \mapsto m_{\lambda^{-1}} \cdot \varphi(m_\lambda \cdot (a, s, p))$$

is an automorphism of  $\mathcal{P}$ . Moreover,  $(\varphi_\lambda)^{-1} = (\varphi^{-1})_\lambda$ ,  $\lambda \in \mathbb{T}$ . Letting  $\lambda \rightarrow 0$  in  $\varphi_\lambda$  and  $\varphi_\lambda^{-1}$  we deduce that the mapping

$$(a, s, p) \mapsto ((s^2 - 4p)\alpha_1(0, 0) + \alpha_2 a, s, \gamma_1 a + p)$$

is an automorphism of the pentablock. Putting  $s = 0$  and making use of the description of the pentablock we deduce that

$$(a, p) \mapsto (-4p\alpha_1(0, 0) + \alpha_2 a, \gamma_1 a + p)$$

is an automorphism of the bidisc. Therefore  $\gamma_1 = \alpha_1(0, 0) = 0$  and  $\alpha_2 = 1$ .

Applying the same reasoning to  $m = (1, 1, 2)$  (it is possible as  $\gamma_1 = 0$  and the same property holds for  $\varphi^{-1}$ ) we find that the following mapping

$$(a, s, p) \mapsto (a, \beta_1 a + s, p + \gamma_3 a s)$$

is an automorphism of  $\mathcal{P}$  for some  $\gamma_3 \in \mathbb{C}$ . In particular, putting  $s = 0$  we see that for any  $a, p \in \mathbb{D}$  the point  $(a, \beta_1 a, p)$  lies in the pentablock. In particular,  $(1, \beta_1, p)$  lies in  $\mathcal{P}$ . This means, that there is  $z \in \mathcal{R}_I$  such that  $\pi(z) = (1, \beta_1, p)$ . Since  $z_{12} = 0$  we see that  $z_{11} = z_{22} = 0$ , whence  $\beta_1 = 0$ .

Step 5. Note that we have shown that  $\varphi'(0) = \text{id}$ . Since  $\varphi$  preserves the origin, the assertion is a direct consequence of the classical Cartan theorem.  $\square$

In [Ag-Ly-Yo] the authors introduced another description of the pentablock which turns out to be useful in the sequel. Namely, the pentablock may be given as the interior of the intersection

$$(4) \quad \mathcal{P} = \text{int} \bigcap_{z \in \mathbb{D}} \{(a, s, p) \in \mathbb{C} \times \mathbb{G}_2 : |\Psi_z(a, s, p)| < 1\},$$

where  $\Psi_z$  is a function given by  $\Psi_z(a, s, p) = \frac{a(1-|z|^2)}{1-sz+pz^2}$ . Note here that  $1-sz+pz^2$  does not vanish provided that  $z \in \mathbb{D}$  and  $(s, p) \in \mathbb{G}_2$ . Using (4) we are able to show the following

**Proposition 3.** *The function  $u$  is  $\mathbb{C}$ -convex.*

Recall here that a real valued  $\mathcal{C}^2$  function  $u$  defined on a subdomain  $D$  of  $\mathbb{C}$  is said to be  $\mathbb{C}$ -convex if the inequality  $u_{z\bar{z}} \geq |u_{zz} - (u_z)^2|^2$  holds on  $D$ . Analogously, a real valued  $\mathcal{C}^2$  function  $u$  of several complex variables is  $\mathbb{C}$ -convex if its restriction to any complex line is  $\mathbb{C}$ -convex in the previous sense. Recall also the following theorem due to Kiselman (see [Kis]):

*a Hartogs domain  $H$  over the unit disc  $H = \{(z, w) \in \mathbb{C} \times \mathbb{D} : |z|^2 < e^{-u(w)}\}$ , where  $u$  is  $\mathcal{C}^2$  smooth, is  $\mathbb{C}$ -convex iff it is weakly linearly convex iff  $u$  is  $\mathbb{C}$ -convex.*

The notion of  $\mathbb{C}$ -convex functions and the above observation were extended by Jacquet to the case when  $u$  is of class  $\mathcal{C}^1$  (see [Jac] for details).

Note that Kiselman's result does not remain true if a disc is replaced by an arbitrary convex domain (for example  $\mathbb{G}_2 \times \mathbb{D}$  is not  $\mathbb{C}$ -convex).

*Proof of Proposition 3.* Take any complex line  $l$  intersecting  $\mathbb{G}_2$  and any disc  $\Delta$  contained in  $l \cap \mathbb{G}_2$ . Since  $u$  is smooth, it suffices to show that the following Hartogs domain  $\{(a, s, p) \in \mathbb{C} \times \Delta : |a|^2 < e^{-u(s, p)}\}$  is linearly convex in  $\mathbb{C} \times l$ .

To do it note that  $\mathcal{P}_z := \{(a, s, p) \in \mathbb{C} \times \mathbb{G}_2 : |\Psi_z(a, s, p)| < 1\}$ , is a Hartogs domain, more precisely

$$\mathcal{P}_z = \{(a, s, p) \in \mathbb{C} \times \mathbb{G}_2 : |a|^2 < e^{-u_z(s, p)}\},$$

where  $u_z(s, p) = -2 \log \left( \frac{1-sz+pz^2}{1-|z|^2} \right)$ ,  $z \in \mathbb{D}$ ,  $(s, p) \in \mathbb{G}_2$ . Clearly,  $u_z$  is  $\mathbb{C}$ -convex, so  $\mathcal{P}_z \cap (\mathbb{C} \times \Delta)$  is a  $\mathbb{C}$ -convex subdomain of  $\mathbb{C} \times l$ , by [Kis]. Therefore,  $\bigcap_{z \in \mathbb{D}} (\mathcal{P}_z \cap (\mathbb{C} \times l))$  is linearly convex. Since the interior of a linearly convex set is linearly convex,  $\mathcal{P} \cap (\mathbb{C} \times \Delta)$  is linearly convex, as well.  $\square$

Note that in the proof of the previous proposition we have actually obtained the following result which is well known for plurisubharmonic and convex functions:

**Proposition 4.** *Let  $\{u_\alpha\}$  be a locally bounded from above family of  $\mathbb{C}$ -convex functions on a domain  $D$  of  $\mathbb{C}^n$ . Let  $u := \sup\{u_\alpha\}$ . If  $u$  is  $\mathcal{C}^1$  smooth, then it is  $\mathbb{C}$ -convex.*

Observe that it follows from Proposition 3 that the pentablock is linearly convex.

Finally we shall prove that the pentablock is not an analytic retract of the open unit ball of any  $\mathcal{J}^*$  algebra of finite rank (see [Har] a definition of a  $\mathcal{J}^*$  algebra). A domain  $P$  is said to be an analytic retract of a domain  $B$  if there are holomorphic mappings  $\varphi : P \rightarrow B$  and  $\psi : B \rightarrow P$  such that  $\psi \circ \varphi = \text{id}$ .

Recall a result of N. Young who showed in [You] that the tetra-block (another domain appearing in the  $\mu$ -synthesis problem) possesses this property. Some analysis of his proof shows that it works for the symmetrized bidisc as well. Actually, the main ingredient of Young's proof is to show that there is no isometry between the unit ball in  $\mathbb{C}^3$  with the norm  $\|x\|_{\mathbb{E}} := \max(|x_1|, |x_2|) + |x_3|$  and the mentioned unit ball in the  $\mathcal{J}^*$  algebra of finite rank. However, the argument used there (and this is actually the core of the idea) shows that such an isometry does not exist if  $\mathbb{C}^3$  equipped with  $\|\cdot\|_{\mathbb{E}}$  will be replaced with  $\mathbb{C}^2$  equipped with the norm  $\|x\|_s = |x_1| + |x_2|$ . On the other the indicatrix of the symmetrized bidisc at the origin is clearly isomorphic with the unit ball in  $(\mathbb{C}^2, \|\cdot\|_s)$ .

Using this we may simply prove that the same property holds for  $\mathcal{P}$ :

**Proposition 5.** *The pentablock is not an analytic retract of the open unit ball of a  $\mathcal{J}^*$  algebra of finite rank.*

*Proof.* Suppose a contrary, i.e there is a  $\mathcal{J}^*$  algebra  $\mathcal{A}$  and there are analytic mappings  $\varphi : \mathcal{P} \rightarrow \mathcal{B}$  and  $\psi : \mathcal{B} \rightarrow \mathcal{P}$  such that  $\psi \circ \varphi = \text{id}$ , where  $\mathcal{B}$  is the unit ball in  $\mathcal{A}$ .

Let us define

$$i : \mathbb{G}_2 \ni (s, p) \mapsto (0, s, p) \in \mathcal{P}$$

and

$$j : \mathcal{P} \ni (a, s, p) \mapsto (s, p) \in \mathbb{G}_2.$$

Then  $\tilde{\varphi} := \varphi \circ i : \mathbb{G}_2 \rightarrow \mathcal{B}$  and  $\tilde{\psi} := j \circ \psi : \mathcal{B} \rightarrow \mathbb{G}_2$  satisfy  $\tilde{\psi} \circ \tilde{\varphi} = \text{id}$ ; a contradiction.  $\square$

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LAVAL UNIVERSITY, QC, CANADA

*E-mail address:* lukasz.kosinski@gazeta.pl